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APPROXIMATION OF INFIMA IN THE CALCULUS OF VARIATIONS

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Abstract : The goal of this paper is to give numerical estimates for some problems of the Calculus of Variations in the nonhomogeneous scalar case. The stored energy function considered is then a function $\varphi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$. We try to compare the infimum of the energy defined by φ on a Sobolev space, with the infimum of the same energy on a finite element space, in terms of the mesh size.

Key words : Calculus of Variations, finite elements, approximation.

Mathematics subject classifications : 26B25, 49XX, 52A20, 65N30, 65N99.

1. Introduction

Let $n \geq 1$ be an integer, Ω be a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary and

$$\varphi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

be a continuous function. Let us define $\overline{\varphi} : W^{1,\infty}(\Omega) \longrightarrow \mathbb{R}$ by

$$\overline{\varphi}(u) = \operatorname*{Inf}_{w \in u + W_0^{1,\infty}(\Omega)} \frac{1}{|\Omega|} \int_\Omega \varphi(x, \nabla w(x))dx.$$

Usually, in various problems of the Calculus of Variations, one is interested in the existence of a function $v \in W^{1,\infty}(\Omega)$ such that

$$\overline{\varphi}(u) = \frac{1}{|\Omega|} \int_\Omega \varphi(x, \nabla v(x))dx \qquad \text{and} \qquad v = u \ \text{ on } \ \partial\Omega.$$

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When we look at $\overline{\varphi}(u)$, we look at a minimization problem where u is a boundary data. But the fact that φ is depending on x allows us to include the boundary data in the function φ : more precisely, if we consider the minimization problem

$$\operatorname*{Inf}_{w\in u+W_0^{1,\infty}(\Omega)} \frac{1}{|\Omega|}\int_\Omega \varphi(x,\nabla w(x))dx \tag{1.1}$$

and if we set $\varphi_u(x,\beta)=\varphi(x,\nabla u(x)+\beta)$ we get the following expression for (1.1)

$$\operatorname*{Inf}_{w\in W_0^{1,\infty}(\Omega)} \frac{1}{|\Omega|}\int_\Omega \varphi_u(x,\nabla w(x))dx.$$

This is the reason why we will just consider in the following the quantity $\overline{\varphi}(0)$.

The aim of this paper is to show that we can approach $\overline{\varphi}(0)$ by an infimum where the Sobolev space $W_0^{1,\infty}(\Omega)$ is replaced by a P_1-finite element space $V_0^h(\Omega)$. We will give more precisely estimates for the difference $\overline{\varphi}^h(0)-\overline{\varphi}(0)$.

Such numerical estimates for problems of the Calculus of Variations - especially for some of which concerning material science and elastic crystals - were considered by many authors ; see, for example, [Br.], [Br.Ch.$_1$], [Br.Ch.$_2$], [Ch.$_1$], [Ch.$_2$], [Ch.$_3$], [Ch.C.], [Ch.C.K.], [Ch.E.], [Ch.L.], [Ch.M.], [C.], [C.K.Lu.], [E.], [L.] and [Lu.].

We will consider the following hypothesis :

(H_1) $\qquad \forall x\in\Omega,\ \forall\beta\in\mathbb{R}^n,\ \ a_1(x)+b_1|\beta|^p\le\varphi(x,\beta)\le a_2(x)+b_2|\beta|^p$

(H_2) $\qquad \forall x,y\in\Omega,\ \ \forall\beta\in\mathbb{R}^n,\ \ |\varphi(x,\beta)-\varphi(y,\beta)|\le\theta_1|x-y|$

(H_3) $\qquad \forall x\in\Omega,\ \forall\beta_1,\beta_2\in\mathbb{R}^n,\ \ |\varphi(x,\beta_1)-\varphi(x,\beta_2)|\le\theta_2|\beta_1-\beta_2|$

where $a_1,a_2\in L^\infty(\Omega)$, $b_2>b_1>0$, $p>1$ and $\theta_1,\theta_2>0$.

Of course, if φ satisfies some of the assumptions (H_1), (H_2) and (H_3), then φ_u does also provided u has some smoothness. See Remark 1.5 below

Remark 1.1 : By definition of $\overline{\varphi}$ it is clear that $\forall u\in W^{1,\infty}(\Omega)$ and $\forall v\in W_0^{1,\infty}(\Omega)$ one has

$$\overline{\varphi}(u+v)=\overline{\varphi}(u).$$

So, $\overline{\varphi}(u)$ depends only on the boundary values of u.

Remark 1.2 : Let $a\in\mathbb{R}^n$, $b\in\mathbb{R}$ and $u\ :\ \Omega\to\mathbb{R}$ such that $u(x)=a.x+b$.

Assume

$$\forall x\in\Omega,\ \forall\beta\in\mathbb{R}^n,\ \ \varphi(x,\beta)=\psi(\beta)$$

then $\overline{\varphi}(u) = \psi^{**}(a)$, where ψ^{**} denotes the convex envelope of ψ, see [D.].

Remark 1.3 : Let us denote by φ^{**} the map of $\Omega \times \mathbb{R}^n$ into $\mathbb{R}$ such that if $x \in \Omega$, $\varphi^{**}(x,.)$ is the convex envelope of $\varphi(x,.)$.

• If (H_1) holds for φ, then it holds for φ^{**} also.
Indeed, it follows from the convexity of the function $\beta \longmapsto a_1(x) + b_1|\beta|^p$.

• If (H_2) holds for φ, then it holds for φ^{**} also.
Indeed, we have, for $x, y \in \Omega$ fixed, that

$$\forall \beta \in \mathbb{R}^n, \quad \varphi(x,\beta) \leq \theta_1|x-y| + \varphi(y,\beta)$$

and thus

$$\varphi^{**}(x,\beta) \leq \theta_1|x-y| + \varphi^{**}(y,\beta).$$

Reversing x and y we get (H_2) for φ^{**}.

• If (H_3) holds for φ, then it holds for φ^{**} also.
Indeed, if $\beta_1, \beta_2 \in \mathbb{R}^n$ and $x \in \Omega$, then

$$\forall \beta \in \mathbb{R}^n, \quad \varphi(x,\beta) \leq \theta_2|\beta_1 - \beta_2| + \varphi(x, \beta + \beta_2 - \beta_1)$$

and therefore

$$\forall \beta \in \mathbb{R}^n, \quad \varphi^{**}(x,\beta) \leq \theta_2|\beta_1 - \beta_2| + \varphi^{**}(x, \beta + \beta_2 - \beta_1).$$

Choosing $\beta = \beta_1$, we get $\varphi^{**}(x,\beta_1) - \varphi^{**}(x,\beta_2) \leq \theta_2|\beta_1 - \beta_2|$ and the result.

Remark 1.4 : If φ is such that (H_1) and (H_2) hold, then

$$\overline{\varphi} = \overline{\varphi^{**}}$$

("Relaxation theorem" : see [D.], Cor. 2.2, ch. 5, § 5.2 p. 235).

Remark 1.5 : Let us consider $u \in W^{1,\infty}(\Omega)$ and define as above $\varphi_u(x,\beta) = \varphi(x, \nabla u(x) + \beta)$. If ∇u is continuous then φ_u is continuous when φ is. Moreover :

• If $u \in W^{1,\infty}(\Omega)$ and if (H_1) holds for φ, then (H_1) holds for φ_u with some other functions a_1' and a_2' (depending on u) and some other constants b_1' and b_2'.
Indeed

$$\begin{aligned}\varphi_u(x,\beta) &= \varphi(x, \nabla u(x) + \beta)\\ &\geq a_1(x) + b_1|\nabla u(x) + \beta|^p\\ &\geq a_1(x) - b_1|\nabla u(x)|^p + 2^{1-p}b_1|\beta|^p\end{aligned}$$

and

$$\varphi_u(x,\beta) \le a_2(x) + b_2|\nabla u(x) + \beta|^p$$

$$\le a_2(x) + 2^{p-1}b_2|\nabla u(x)|^p + 2^{p-1}b_2|\beta|^p.$$

• If $u \in W^{2,\infty}(\Omega)$, if (H_2) and (H_3) hold for φ, then (H_2) holds for φ_u with some other constant θ_1' (depending on u).
Indeed, it follows from (H_2) and (H_3) that

$$|\varphi_u(x,\beta) - \varphi_u(y,\beta)| = |\varphi(x,\nabla u(x) + \beta) - \varphi(y,\nabla u(y) + \beta)|$$

$$\le |\varphi(x,\nabla u(x) + \beta) - \varphi(x,\nabla u(y) + \beta)| + |\varphi(x,\nabla u(y) + \beta) - \varphi(y,\nabla u(y) + \beta)|$$

$$\le \theta_2|\nabla u(x) - \nabla u(y)| + \theta_1|x-y|$$

$$\le \theta_2'|x-y| + \theta_1|x-y|$$

(since $u \in W^{2,\infty}(\Omega)$)

$$\le \theta_1'|x-y|.$$

• If (H_3) holds for φ, then (H_3) holds for φ_u (with the same constant θ_2).
It is easy :

$$|\varphi_u(x,\beta_1) - \varphi_u(x,\beta_2)| = |\varphi(x,\nabla u(x) + \beta_1) - \varphi(x,\nabla u(x) + \beta_2)| \le \theta_2|\beta_1 - \beta_2|.$$

• Finally, let us prove that

$$\forall x \in \Omega,\ \forall \beta \in \mathbb{R}^n,\ \ (\varphi_u)^{**}(x,\beta) = \varphi^{**}(x,\nabla u(x) + \beta) \tag{1.2}$$

Indeed, let $x \in \Omega$ be fixed. The function $\beta \longmapsto \varphi^{**}(x,\nabla u(x) + \beta)$ is convex and verifies

$$\forall \beta \in \mathbb{R}^n\ ,\ \ \varphi^{**}(x,\nabla u(x) + \beta) \le \varphi(x,\nabla u(x) + \beta) = \varphi_u(x,\beta)$$

implying that

$$\varphi^{**}(x,\nabla u(x) + .) \le (\varphi_u)^{**}(x,.).$$

Conversely, the function $\beta \longmapsto (\varphi_u)^{**}(x,\beta - \nabla u(x))$ is convex and verifies

$$\forall \beta \in \mathbb{R}^n\ ,\ \ (\varphi_u)^{**}(x,\beta - \nabla u(x)) \le \varphi_u(x,\beta - \nabla u(x)) = \varphi(x,\beta)$$

implying that

$$\forall \beta \in \mathbb{R}^n\ ,\ \ (\varphi_u)^{**}(x,\beta - \nabla u(x)) \le \varphi^{**}(x,\beta)$$

and thus

$$\forall \beta \in \mathbb{R}^n \ , \quad (\varphi_u)^{**}(x,\beta) \leq \varphi^{**}(x, \nabla u(x) + \beta).$$

So (1.2) is proved.

The following result gives some useful properties of $\overline{\varphi}$:

Theorem 1.1 : *If φ satisfies (H_1) and (H_2), then the function $\overline{\varphi} \ : \ W^{1,\infty}(\Omega) \longrightarrow \mathbb{R}$ is convex and locally Lipschitz.*

Proof : First, let us prove that $\overline{\varphi}$ is convex.

Let $u, v \in W^{1,\infty}(\Omega)$ and $\epsilon > 0$. By definition of $\overline{\varphi}$ and since $\overline{\varphi} = \overline{\varphi^{**}}$ (see Remark 1.5), there exist $u_\epsilon, v_\epsilon \in W_0^{1,\infty}(\Omega)$ such that

$$\overline{\varphi}(u) \geq \frac{1}{|\Omega|}\int_\Omega \varphi^{**}(x, \nabla u(x) + \nabla u_\epsilon(x))dx \ - \epsilon$$

and

$$\overline{\varphi}(v) \geq \frac{1}{|\Omega|}\int_\Omega \varphi^{**}(x, \nabla v(x) + \nabla v_\epsilon(x))dx \ - \epsilon.$$

Then, since φ^{**} is convex, one has for $\lambda \in [0,1]$

$$\lambda\overline{\varphi}(u) + (1-\lambda)\overline{\varphi}(v) \geq \frac{1}{|\Omega|}\int_\Omega \Big[\lambda\varphi^{**}(x, \nabla u(x) + \nabla u_\epsilon(x))$$

$$+ (1-\lambda)\varphi^{**}(x, \nabla v(x) + \nabla v_\epsilon(x))\Big] dx - \epsilon$$

$$\geq \frac{1}{|\Omega|}\int_\Omega \varphi^{**}(x, \lambda\nabla u(x) + (1-\lambda)\nabla v(x) + \nabla w_\epsilon(x))dx - \epsilon$$

$$\text{where} \quad w_\epsilon = \lambda u_\epsilon + (1-\lambda)v_\epsilon \in W_0^{1,\infty}(\Omega).$$

Thus

$$\lambda\overline{\varphi}(u) + (1-\lambda)\overline{\varphi}(v) \geq \overline{\varphi}(\lambda u + (1-\lambda)v) - \epsilon$$

and this inequality being true for all $\epsilon > 0$, we obtain the convexity of $\overline{\varphi}$.

Now, let us prove that $\overline{\varphi}$ is locally Lipschitz.

For all function $v \in W_0^{1,\infty}(\Omega)$ the map

$$\Psi_v \ : \ u \longmapsto \frac{1}{|\Omega|} \int_\Omega \varphi(x, \nabla u(x) + \nabla v(x)) dx$$

is continuous on $W^{1,\infty}(\Omega)$, in such way that

$$\overline{\varphi} = \operatorname*{Inf}_{v \in W_0^{1,\infty}(\Omega)} \Psi_v$$

is upper semicontinuous, and so $\forall u \in W^{1,\infty}(\Omega)$ there exist a neighbourhood of u where $\overline{\varphi}$ is bounded from above. Consequently, since $\overline{\varphi}$ is convex, we deduce that $\overline{\varphi}$ is locally Lipschitz on $W^{1,\infty}(\Omega)$, (see [E.T.] Cor. 2.4, ch.I p. 12). □

2. Approximation

From now on, we will assume that Ω is a polygonal domain. Then, let $\{\mathcal{T}_h \ : \ h > 0\}$ be a family of regular triangulation of Ω (see [R.T.]), that is to say satisfying

$$\forall h > 0 \qquad \begin{cases} \forall K \in \mathcal{T}_h, \ K \text{ is a } n\text{-simplex} \\ \\ \max\limits_{K \in \mathcal{T}_h} (h_K) = h \\ \\ \forall K \in \mathcal{T}_h \ , \ \ \dfrac{h_K}{\rho_K} \leq \nu \quad (\nu > 0) \end{cases} \tag{2.1}$$

where h_K is the diameter of the n-simplex K and ρ_K its roundness (i.e. the largest diameter of the balls that could fit into K).

If $P_1(K)$ is the space of polynomials of degree 1 on K, we set

$$V^h(\Omega) = \{v \ : \ \Omega \to \mathbb{R} \ \text{ continuous } : \ v|_K \in P_1(K), \ \forall K \in \mathcal{T}_h \ \}$$

and

$$V_0^h(\Omega) = \{v \in V_h(\Omega) \ : \ \ v = 0 \text{ on } \partial\Omega\}.$$

Then, we define $\overline{\varphi}^h \ : \ W^{1,\infty}(\Omega) \longrightarrow \mathbb{R}$ by

$$\forall u \in W^{1,\infty}(\Omega), \ \ \overline{\varphi}^h(u) = \operatorname*{Inf}_{w \in u + V_0^h(\Omega)} \frac{1}{|\Omega|} \int_\Omega \varphi(x, \nabla w(x)) dx.$$

Now, our goal is to get estimates of $\overline{\varphi}^h(u) - \overline{\varphi}(u)$ (we know that $\overline{\varphi}^h(u) \geq \overline{\varphi}(u)$ since $W_0^{1,\infty}(\Omega)$ contains $V_0^h(\Omega)$) and in particular of $\overline{\varphi}^h(0) - \overline{\varphi}(0)$; see §1.

Remark 2.1 : It follows from (2.1) that, if $v \in W^{1,\infty}(\Omega)$, the function $v_h \in V^h(\Omega)$. that interpolates v on $\mathcal{T}_h$, satisfies

$$|\nabla v_h(x)| \leq C_0 |\nabla v(x)| \quad \text{a.e. in } \Omega \tag{2.2}$$

for some constant C_0 depending only on n and ν. See [Br.Ch.$_2$], Lemma 2.1.

Remark 2.2 : If $u \in W^{1,\infty}(\Omega)$ and $v \in V_0^h(\Omega)$, one has $\overline{\varphi}^h(u+v) = \overline{\varphi}^h(u)$.

Remark 2.3 : If $\forall x \in \Omega$, $\forall \beta \in \mathbb{R}^n$, $\varphi(x,\beta) = \psi(\beta)$, and $u(x) = a.x + b$ $(a \in \mathbb{R}^n, b \in \mathbb{R})$, see Remark 1.2, one will find estimates of $\overline{\varphi}^h(u) - \overline{\varphi}(u)$ in [Br.], [Br.Ch$_1$], [Br.Ch$_2$].

3. A preliminary estimate

For a set $A \subset \mathbb{R}^n$, we denote by $co(A)$ the convex hull of A.

In this part, we will use some "saw-tooth" functions constructed in [Ch.$_1$] ; let us recall that :

Lemma 3.1 : *Let $\omega_0, \ldots, \omega_k \in \mathbb{R}^n$ be such that the dimension of the vector space W spanned by $\omega_1 - \omega_0, \ldots, \omega_k - \omega_0$ is equal to k and such that $0 \in co\{\omega_0, \ldots, \omega_k\}$. For each $\eta > 0$ there exists a piecewise affine function $w : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying*

$$|w(x)| \leq \eta \qquad \forall x \in \mathbb{R}^n \tag{3.1}$$

$$\nabla w(x) \in \{\omega_0, \ldots, \omega_k\} \qquad a.e. \text{ in } \mathbb{R}^n \tag{3.2}$$

and if we denote by S the subset of $\mathbb{R}^n$ where w has no derivative, then, for any bounded domain $D \subset \mathbb{R}^n$, we have that

$$\textit{the } (n-1)\textit{-dimensional measure of } D \cap S \textit{ is less than } C_1 |D| \eta^{-1} \tag{3.3}$$

where C_1 is a constant only depending on $\omega_0, \ldots, \omega_k$.

Proof : See [Ch.$_1$], especially the proof of Theorem 1.

Theorem 3.1 : *Let* $\psi \ : \ \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *satisfying*

- ψ *is nonnegative* $\hfill (3.4)$
- *for almost all* $x \in \Omega$, $\exists\ \omega_0(x), \dots, \omega_n(x)$ *(not necessarily two by two distinct) such that*

$$\begin{cases} \text{(i)} & 0 \in co\{\omega_0(x), \dots, \omega_n(x)\} \\ \text{(ii)} & \forall i = 0, \dots, n \ \ \psi(x, \omega_i(x)) = 0 \\ \text{(iii)} & \exists K > 0 \text{ such that } \forall x \in \Omega,\ \forall i = 0, \dots, n \ \ |\omega_i(x)| \leq K \end{cases} \tag{3.5}$$

- *there exist* $\kappa_1, \kappa_2 \geq 0$ *and* $p > 1$ *such that* $\forall \beta \in \mathbb{R}^n$, $\forall x, y \in \Omega$ *one has*

$$|\psi(x,\beta) - \psi(y,\beta)| \leq \kappa_1 |x - y|(1 + \kappa_2 |\beta|) \tag{3.6}$$

Then $\forall h > 0$, $\exists u_h \in V_0^h(\Omega)$ *such that*

$$|u_h|_{L^\infty(\Omega)} \leq h^{\frac{2}{3}} \tag{3.7}$$

and

$$\frac{1}{|\Omega|} \int_\Omega \psi(x, \nabla u_h(x)) dx \ \ \leq C h^{\frac{1}{3}} \tag{3.8}$$

where C *is a constant depending on* $\psi, \Omega, K, \kappa_1, \kappa_2$ *and* p.

In particular, $0 \leq \overline{\psi}^h(0) \leq C h^{\frac{1}{3}}$.

Proof : Let us denote by γ and δ some real numbers such that $0 < \gamma < \delta < 1$ (remark that for $h << 1$, we have $h << h^\delta << h^\gamma$).

Let us cover Ω by n-cubes $Q_1, \dots, Q_s$, $Q'_1, \dots, Q'_r$ of side h^γ where

$$\forall i = 1, \dots, s \quad Q_i \subset \Omega \qquad \text{and} \qquad \forall j = 1, \dots, r \quad Q'_j \cap \partial\Omega \neq \emptyset.$$

For each $i \in \{1, \dots, s\}$ let us consider $x_i \in Q_i$ satisfying (3.5). Whithout lost of generality, we can assume, if the dimension of the vector space W_i spanned by $\omega_1(x_i) - \omega_0(x_i), \dots, \omega_n(x_i) - \omega_0(x_i)$ is equal to k_i, that $\omega_1(x_i) - \omega_0(x_i), \dots, \omega_{k_i}(x_i) - \omega_0(x_i)$ are linearly independant.

Now, let us consider the function w_i of the Lemma 3.1 corresponding to $\eta = h^\delta$ and $D = Q_i$, and denote by v_i the function defined on Q_i by

$$x \longmapsto \max\big[\min\big(\mathrm{d}(x, \partial Q_i)\,;\ w_i(x)\big)\,;\ -\mathrm{d}(x, \partial Q_i)\big].$$

Since $v_i = 0$ on ∂Q_i, we can define on Ω a continuous function u equal to v_i on Q_i and to 0 on Q'_j. Then, let us denote by u_h the $\mathcal{T}_h$-interpolate of u.

First, $u_h \in V_0^h(\Omega)$. Secondly, it follows from the definition of u and (3.1) that

$$|u_h(x)| \leq h^\delta \quad \text{a.e. in} \quad \Omega. \tag{3.9}$$

Third, by (3.1), (3.2), (2.2) and (3.5) (iii) we get that

$$|\nabla u_h(x)| \leq C_0 |\nabla u(x)| \leq C_2 \quad \text{a.e. in} \quad \Omega. \tag{3.10}$$

To obtain an estimate of

$$\int_\Omega \psi(x, \nabla u_h(x)) dx$$

let us write

$$\int_\Omega \psi(x, \nabla u_h(x)) dx \leq \sum_{i=1}^s \int_{Q_i} \Big| \psi(x, \nabla u_h(x)) - \psi(x_i, \nabla u_h(x)) \Big| dx$$

$$+ \sum_{i=1}^s \int_{Q_i} \psi(x_i, \nabla u_h(x)) dx + \int_{\Omega_B} \psi(x, \nabla u_h(x)) dx \tag{3.11}$$

where $\Omega_B = \Omega \cap (Q'_1 \cup \ldots \cup Q'_r)$, and consider separately each of the terms in the right hand side of this inequality.

- By using (3.6) and (3.10) we have that

$$\sum_{i=1}^s \int_{Q_i} \Big| \psi(x, \nabla u_h(x)) - \psi(x_i, \nabla u_h(x)) \Big| dx \leq \sum_{i=1}^s \int_{Q_i} \kappa_1 |x - x_i| (1 + \kappa_2 |\nabla u_h(x)|) dx$$

$$\leq \kappa_1 h^\gamma (1 + \kappa_2 C_2) \sum_{i=1}^s |Q_i|$$

$$\leq C_3 |\Omega| h^\gamma. \tag{3.12}$$

- Since $|\Omega_B| \leq C_4 |\partial\Omega| h^\gamma$, we have from (3.4) and (3.10) that

$$\int_{\Omega_B} \psi(x, \nabla u_h(x)) dx \leq C_5 h^\gamma. \tag{3.13}$$

- For each $i \in \{1, \ldots, s\}$, we have that

$$\nabla u_h(x) \in \{\omega_0(x_i), \ldots, \omega_{k_i}(x_i)\} \quad \text{a.e. in} \quad Q_i$$

except, perhaps, on a neighbourhood $S_{i,1}$ of ∂Q_i such that

$$|S_{i,1}| \leq C_6|\partial Q_i|(h^\delta + h) \leq C_7|Q_i|h^{\delta-\gamma} \tag{3.14}$$

and on the set $S_{i,2}$ made of the n-simplices of $\mathcal{T}_h$ which intersect $S_i \cap Q_i$ (with, as in the Lemma 3.1, S_i the subset of $\mathbb{R}^n$ where w_i has no derivative) ; by definition of $S_{i,2}$ and (3.3) we have that

$$|S_{i,2}| \leq C_8\frac{|Q_i|}{h^\delta} \cdot h = C_8|Q_i|h^{1-\delta}. \tag{3.15}$$

Hence, we have from (3.6), (3.10), (3.14) and (3.15) that

$$\begin{aligned}\sum_{i=1}^{s}\int_{Q_i}\psi(x_i,\nabla u_h(x))dx &= \sum_{i=1}^{s}\int_{S_{i,1}\cup S_{i,2}}\psi(x_i,\nabla u_h(x))dx \\ &\leq C_9\sum_{i=1}^{s}(|S_{i,1}|+|S_{i,2}|) \\ &\leq C_9\left(\sum_{i=1}^{s}|Q_i|\right)(C_7h^{\delta-\gamma}+C_8h^{1-\delta}) \\ &\leq C_{10}|\Omega|(h^{\delta-\gamma}+h^{1-\delta}) \end{aligned}\tag{3.16}$$

Finally, using (3.11), (3.12), (3.13) and (3.16) we get

$$\begin{aligned}\int_\Omega \psi(x,\nabla u_h(x))dx &\leq C_{11}|\Omega|(h^\gamma + h^{\delta-\gamma} + h^{1-\delta}) \\ &\leq 3C_{11}|\Omega|h^{\min(\gamma,\delta-\gamma,1-\delta)}\end{aligned}$$

So, by choosing $\gamma = \frac{1}{3}$ and $\delta = \frac{2}{3}$, we obtain (3.8), and (3.9) give (3.7). □

Remark 3.1 : To compute numerically

$$\int_\Omega \psi(x,\nabla u_h(x))dx$$

it seems to be natural to consider

$$\sum_{K\in\mathcal{T}_h}|K|\psi(x_K,\beta_K)dx$$

where $\beta_K = \nabla u_{h|K}$ and x_K is the center of K. If ψ satisfies (3.6), we have that

$$\left| \int_\Omega \psi(x, \nabla u_h(x))dx - \sum_{K \in \mathcal{T}_h} |K| \psi(x_K, \beta_K) dx \right| \le Ch.$$

4. Estimate of $\overline{\varphi}^h(0) - \overline{\varphi}(0)$

First, let us recall a geometrical lemma :

Lemma 4.1 : *Suppose that $f \ : \ \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous and that*

$$\lim_{|\beta| \to +\infty} \frac{f(\beta)}{|\beta|} = +\infty.$$

Let $\alpha \in \mathbb{R}^n$. Then, there exist $\alpha_0, \ldots, \alpha_n \in (f - f^{**})^{-1}(0)$ not necessarily two by two distinct such that

$$\alpha \in co\{\alpha_0, \ldots, \alpha_n\} \quad and \quad f^{**} \quad is\ affine\ on \quad co\{\alpha_0, \ldots, \alpha_n\}$$

($co\{\alpha_0, \ldots, \alpha_n\}$ could be reduce to $\{\alpha\}$).

Proof : See [Br.Ch.$_2$].

Theorem 4.1 : *Let us assume that φ verifies (H_1) and (H_2) and that*

$$\overline{\varphi}(0) = \mathop{Inf}_{w \in W_0^{1,\infty}(\Omega)} \frac{1}{|\Omega|} \int_\Omega \varphi(x, \nabla w(x))dx \ = \frac{1}{|\Omega|} \int_\Omega \varphi^{**}(x, 0)dx. \tag{4.1}$$

Moreover, suppose that $\forall x \in \Omega,\ \exists A(x) \in \partial\varphi^{**}(x, 0)$ verifying for some constant C_1

$$|A(x) - A(y)| \le C_1 |x - y| \ , \quad \forall x, y \in \Omega. \tag{4.2}$$

Then, $\forall h > 0$,

$$0 \le \overline{\varphi}^h(0) - \overline{\varphi}(0) \le Ch^{1/3}$$

where C is a constant depending on φ, Ω, C_1 and the constants appearing in (H_1) and (H_2).

Proof : (Let us recall that $\partial\varphi^{**}(x, 0)$ denotes the subdifferential of $\varphi^{**}(x, .)$ at the point 0.)

Let $x \in \Omega$. Thanks to (H_1) one has

$$\lim_{|\beta|\to+\infty} \frac{\varphi(x,\beta)}{|\beta|} = +\infty.$$

Thus, from the continuity of φ and the lemma 4.1, there exist $\alpha_0(x), \dots, \alpha_n(x) \in \mathbb{R}^n$ (not necessarily pairwise distinct) such that

$$\forall i = 0, \dots, n \quad \varphi(x, \alpha_i(x)) = \varphi^{**}(x, \alpha_i(x)) \tag{4.3}$$

$$0 \in co\{\alpha_0(x), \dots, \alpha_n(x)\} \tag{4.4}$$

$$\varphi^{**}(x, .) \text{ is affine on } co\{\alpha_0(x), \dots, \alpha_n(x)\}. \tag{4.5}$$

Next, set

$$\begin{aligned} g \ : \ \Omega \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x, \beta) &\longmapsto A(x).\beta + \varphi^{**}(x, 0). \end{aligned}$$

Let us remark that we have in particular

$$g(x, 0) = \varphi^{**}(x, 0). \tag{4.6}$$

Now, we are going to prove that one can use the Theorem 3.1 for the function $\psi = \varphi - g$.

• First, from the continuity of φ and the definition of g, we have that $\forall x \in \Omega$, $\psi(x, .)$ is continuous.

• Second, $\psi \geq 0$; indeed $\forall x \in \Omega$, $\forall \beta \in \mathbb{R}^n$ one has

$$\begin{aligned} \psi(x, \beta) &= \varphi(x, \beta) - g(x, \beta) \\ &\geq \varphi(x, \beta) - \varphi^{**}(x, \beta) \end{aligned}$$

since $A(x) \in \partial\varphi^{**}(x, 0)$ means that

$$\forall \beta \in \mathbb{R}^n, \quad A(x).\beta \leq \varphi^{**}(x, \beta) - \varphi^{**}(x, 0). \tag{4.7}$$

Thus $\psi(x, \beta) \geq 0$.

• Third, $\forall x \in \Omega$, one has (4.4) and $\psi(x, \alpha_i(x)) = \varphi(x, \alpha_i(x)) - g(x, \alpha_i(x))$; but from (4.4) and (4.5) we deduce that $g(x, .)$ and $\varphi^{**}(x, .)$ coïncide on $co\{\alpha_0(x), \dots, \alpha_n(x)\}$ (indeed, from (4.7) one has $\varphi^{**}(x, .) \geq g(x, .)$ and if $g(x, \alpha_j(x)) < \varphi^{**}(x, \alpha_j(x))$ for some j, then

$$g(x, 0) = \sum_{i=0}^{n} \lambda_i g(x, \alpha_i(x)) < \sum_{i=0}^{n} \lambda_i \varphi^{**}(x, \alpha_i(x)) = \varphi^{**}(x, 0)$$

and this is a contradiction with (4.6). Note that, without lost of generality, we can assume that $\lambda_i > 0$.)

Consequently, by (4.3) we have that

$$\psi(x, \alpha_i(x)) = \varphi(x, \alpha_i(x)) - \varphi^{**}(x, \alpha_i(x)) = 0$$

• Next, $\forall x, y \in \Omega$, $\forall \beta \in \mathbb{R}^n$ one has that :

$$\begin{aligned} |\psi(x,\beta) - \psi(y,\beta)| &\leq |\varphi(x,\beta) - \varphi(y,\beta)| + |g(x,\beta) - g(y,\beta)| \\ &\leq \theta_1|x-y| + |A(x) - A(y)|.|\beta| + |\varphi^{**}(x,0) - \varphi^{**}(y,0)|. \end{aligned}$$

By using (4.2) and the Remark 1.3, we get

$$\begin{aligned} |\psi(x,\beta) - \psi(y,\beta)| &\leq 2\theta_1|x-y| + C_1|x-y|\ |\beta| \\ &= 2\theta_1|x-y|(1 + \frac{C_1}{2\theta_1}\ |\beta|). \end{aligned}$$

• Finally, it remains to show that there exists a constant $K > 0$ independent of x, such that

$$\forall i = 0, \ldots, n,\ \forall x \in \Omega,\ |\alpha_i(x)| \leq K. \tag{4.8}$$

We know that $\forall x \in \Omega$, $\forall \beta \in \mathbb{R}^n$, one has from (H_1) and the Remark 1.3

$$a_1(x) + b_1|\beta|^p \leq \varphi^{**}(x,\beta) \leq a_2(x) + b_2|\beta|^p$$

thus, for $\beta \neq 0$,

$$\frac{a_1(x)}{|\beta|} + b_1|\beta|^{p-1} \leq \frac{\varphi^{**}(x,\beta)}{|\beta|}.$$

Therefore

$$\frac{\varphi^{**}(x,\beta)}{|\beta|} \geq b_1|\beta|^{p-1} - \frac{|a_1|_{L^\infty(\Omega)}}{|\beta|}$$

and

$$\frac{\varphi^{**}(x,\beta) - \varphi^{**}(x,0)}{|\beta|} \to +\infty \ \text{ uniformly with respect to } x, \text{ when } |\beta| \to +\infty.$$

Since the function $A : \Omega \to \mathbb{R}^n$ is Lipschitz, and Ω bounded, one has $A \in L^\infty(\Omega, \mathbb{R}^n)$. Then, there exists $K > 0$ such that

$$|\beta| > K \Longrightarrow \varphi^{**}(x,\beta) - \varphi^{**}(x,0) > |A|_{L^\infty}.|\beta| \ , \quad \forall x \in \Omega. \tag{4.9}$$

Let us suppose that there exist $x \in \Omega$ and $i \in \{0, \ldots, n\}$ such that $|\alpha_i(x)| > K$. then by (4.9) we have that

$$A(x).\alpha_i(x) = \varphi^{**}(x, \alpha_i(x)) - \varphi^{**}(x,0) > |A|_{L^\infty}.|\alpha_i(x)| \tag{4.10}$$

but, using the Cauchy-Schwarz inequality, we get

$$|A|_{L^\infty}.|\alpha_i(x)| \geq |A(x)|.|\alpha_i(x)| \geq |A(x).\alpha_i(x)|$$

thus (4.10) implies $A(x).\alpha_i(x) > |A(x).\alpha_i(x)|$ which is impossible.

Consequently (4.8) holds.

Now, we can apply the Theorem 3.1. Therefore, $\forall h > 0$, $\exists\ u_h \in V_0^h(\Omega)$ such that

$$|u_h|_{L^\infty(\Omega)} \leq h^{\frac{2}{3}} \quad \text{and} \quad \frac{1}{|\Omega|}\int_\Omega \psi(x, \nabla u_h(x))dx \ \leq C_2 h^{\frac{1}{3}}$$

(where C_2 is independent of h).

Then

$$\begin{aligned}\overline{\varphi}^h(0) &\leq \frac{1}{|\Omega|}\int_\Omega \psi(x, \nabla u_h(x))dx \ + \frac{1}{|\Omega|}\int_\Omega \ g(x, \nabla u_h(x))dx \\ &\leq C_2 h^{\frac{1}{3}} + \frac{1}{|\Omega|}\int_\Omega A(x).\nabla u_h(x)dx \ + \frac{1}{|\Omega|}\int_\Omega \varphi^{**}(x,0)dx\end{aligned}$$

and, since the second integral in the last inequality is equal to $\overline{\varphi}(0)$, one has

$$\overline{\varphi}^h(0) - \overline{\varphi}(0) \leq C_2 h^{\frac{1}{3}} + \frac{1}{|\Omega|}\int_\Omega A(x).\nabla u_h(x)dx \tag{4.11}$$

Next, since $A = (A_i) \ : \ \Omega \longrightarrow \mathbb{R}^n$ is Lipschitz, for all $\Omega' \subset\subset \Omega$, all $s \in \mathbb{R}^n$ such that $|s| < d(\Omega', \partial\Omega)$ and all $x \in \Omega'$, one has

$$|A_i(x+s) - A_i(x)| \leq C_1|s|$$

and thus $A_i \in W^{1,\infty}(\Omega)$ and $|\nabla A_i|_{L^\infty(\Omega)} \leq C_1$ (see [B.] Prop IX.3 p. 153).

Therefore, we can write

$$\left| \int_\Omega A(x).\nabla u_h(x)dx \right| = \left| \int_\Omega \text{div } A(x) \; u_h(x)dx \right| \quad \text{since } u_h = 0 \text{ on } \partial\Omega$$

$$\leq |u_h|_{L^\infty} \int_\Omega |\text{div } A(x)|dx$$

$$\leq nC_1 \; |\Omega| h^{\frac{2}{3}}.$$

and thanks to (4.11) we obtain

$$\overline{\varphi}^h(0) - \overline{\varphi}(0) \leq Ch^{\frac{1}{3}}$$

where C is a constant depending on φ, Ω, C_1 and on the constants from (H_1) and (H_2). Now, the proof is complete. □

Remark 4.1 : • In the previous theorem, the hypothesis (4.1) is satisfied in the following case :

$$\forall x \in \Omega, \; \forall \beta \in \mathbb{R}^n, \quad \varphi(x,\beta) \geq \varphi^{**}(x,0). \tag{4.12}$$

Indeed, if (4.12) holds, then for $v \in W_0^{1,\infty}(\Omega)$, we have that

$$\int_\Omega \varphi(x,\nabla v(x))dx \geq \int_\Omega \varphi^{**}(x,0)dx \geq \operatorname*{Inf}_w \int_\Omega \varphi^{**}(x,\nabla w(x))dx = \operatorname*{Inf}_w \int_\Omega \varphi(x,\nabla w(x))dx$$

and (4.1) follows.

Moreover, if $\varphi(x,\beta) = f(x)\psi(\beta)$ with $f \geq 0$ and

$$\forall \beta \in \mathbb{R}^n, \quad \psi(\beta) \geq \psi^{**}(0). \tag{4.13}$$

then (4.12) holds. (Remark that, for example, the function $\psi(\beta) = (|\beta|^2 - a^2)^2$ verifies (4.13).)

• The hypothesis (4.2) holds, for instance, if $\forall x \in \Omega$, $\varphi^{**}(x,.)$ has derivative at 0 and if the function $x \longmapsto \dfrac{\partial \varphi^{**}}{\partial \beta}(x,0)$ is Lipschitz.

Theorem 4.2 : *Let us assume that the function φ satisfies (H_1), (H_2) and (H_3), and*

$$\overline{\varphi}(0) = \mathop{Inf}_{w \in W_0^{1,\infty}(\Omega)} \frac{1}{|\Omega|} \int_\Omega \varphi(x,\nabla w(x))dx \; = \frac{1}{|\Omega|} \int_\Omega \varphi^{**}(x,\nabla a(x))dx \tag{4.14}$$

where $a \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$.

Moreover, suppose that $\forall x \in \Omega,\ \exists A(x) \in \partial\varphi^{**}(x, \nabla a(x))$ *verifying for some constant* C_1

$$\forall x, y \in \Omega, \quad |A(x) - A(y)| \leq C_1 |x - y|. \tag{4.15}$$

Then, $\forall h > 0$,

$$0 \leq \overline{\varphi}^h(0) - \overline{\varphi}(0) \leq C h^{\frac{1}{3}}$$

where C *is a constant depending on* φ, Ω, C_1*, on the function* a *and on the constants from* (H_1), (H_2) *and* (H_3).

Proof : In order to use the previous theorem, let us introduce the following function $\varphi_a\ :\ \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$\forall x \in \Omega \quad \text{and} \quad \forall \beta \in \mathbb{R}^n\ ,\ \ \varphi_a(x, \beta) = \varphi(x, \nabla a(x) + \beta).$$

We would like to show that the function φ_a satisfies the assumptions of the theorem 4.1. First, we have that

$$\begin{aligned}
\overline{\varphi}_a(0) &= \overline{\varphi}(a) \\
&= \overline{\varphi}(0) \text{ since } \ a \in W_0^{1,\infty}(\Omega) \quad \text{(see Remark 1.1)} \\
&= \frac{1}{|\Omega|} \int_\Omega \varphi^{**}(x, \nabla a(x)) dx \quad \text{by (4.14)} \\
&= \frac{1}{|\Omega|} \int_\Omega (\varphi_a)^{**}(x, 0) dx
\end{aligned}$$

where the last equality results from (1.2).

Now, it is clear that $\forall x \in \Omega$, $A(x) \in \partial(\varphi_a)^{**}(x, 0)$ and thus, by (4.15) we have (4.2) for the function φ_a.

Finally, it remains to use the Remark 1.5 to see that (H_1), (H_2) and (H_3) hold for φ_a. To conclude, we apply the theorem 4.1 ; and this completes the proof. □

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